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Derivation of Black-Scholes equation from Binomial Model

From N-step Binomial model, we have

$$C_0 = e^{-rT} \sum_{i=0}^{N} Max(0, S_N - X) \frac{N!}{i!\,(N-i)!} p^i\, q^{N-i} \qquad \text{Eq. 1}$$

Where

C_0 : the price of the call option
X : the strike price of the option
r : the risk free interest rate; the rate on a safe asset with the same maturity as the option
T : time till the maturity of the option
N : the number of steps in the model
$S_N = S_0\, u^i\, d^{(N-i)}$: the price of the stock at step N
S_0 : the current price of the underlying stock
$u = e^{\sigma\sqrt{\Delta t}}$
$d = u^{-1}$
$p = \dfrac{e^{r\Delta t} - d}{u - d}$
$q = 1 - p$
σ : standard deviation of the annualized continuously compounded rate of return on the stock

Call option will be in the money (that is, will still have value) when

$$S_0\, u^k\, d^{N-k} \geq X$$

From which we deduce

$$u^{2k-N} \geq X / S_0$$

$$k \geq \left[\frac{\ln(X/S_0)}{2\ln(u)} + \frac{N}{2} \right] = \frac{\ln(X/S_0)}{2\sigma\sqrt{\Delta t}} + \frac{N}{2} \qquad \text{Eq. 2}$$

Note that k is not necessarily an integer.

We can rewrite C_0 as

$$C_0 = e^{-rT} \sum_{i=k}^{N} (S_0 u^i d^{N-i} - X) \frac{N!}{i!\,(N-i)!} p^i q^{N-i} \qquad \text{Eq. 3}$$

Where k is the first integer satisfying Eq.2.

There are two sums to handle:

$$I_1 = \sum_{i=k}^{N} u^i d^{N-i} \frac{N!}{i!(N-i)!} p^i q^{N-i}$$

And

$$I_2 = \sum_{i=k}^{N} \frac{N!}{i!(N-i)!} p^i q^{N-i}$$

With these definitions Eq.3 reads as

$$C_0 = e^{-rT} (S_0 I_1 - X I_2)$$

Let's first look at I_2; by using de Moivre-Laplace Theorem:

$$\frac{N!}{i!(N-i)!} p^i q^{N-i} \approx \frac{1}{\sqrt{2\pi Npq}} e^{-(i-Np)^2/2Npq} \qquad \text{Eq. 4}$$

(See http://mathworld.wolfram.com/deMoivre-LaplaceTheorem.html for details.)

This theorem follows from Stirling approximation, which is

$$N! \approx \sqrt{2\pi N}\, N^N e^{-N}$$

$$I_2 = \sum_{i=k}^{N} \frac{N!}{i!(N-i)!} p^i q^{N-i} \approx \sum_{i=k}^{N} \frac{1}{\sqrt{2\pi Npq}} e^{-(i-Np)^2/2Npq}$$

Now let's take the limit

$$N \Delta t \underset{\substack{N\to\infty \\ \Delta t \to 0}}{\longrightarrow} T$$

$$p=\frac{e^{r\Delta t}-d}{u-d}\approx\frac{(1+r\Delta t+r^2\Delta t^2/2+\cdots)-(1-\sigma\sqrt{\Delta t}+\sigma^2\Delta t/2-\cdots)}{(1+\sigma\sqrt{\Delta t}+\sigma^2\Delta t/2+\cdots)-(1-\sigma\sqrt{\Delta t}+\sigma^2\Delta t/2+\cdots)}$$
$$=(r\Delta t+\sigma\sqrt{\Delta t}-\sigma^2\Delta t/2)/(2\sigma\sqrt{\Delta t})$$
$$=\frac{1}{2}+\frac{1}{2}\frac{(r-\sigma^2/2)}{\sigma}\sqrt{\Delta t} \qquad \text{Eq. 5a}$$

And

$$q=1-p\approx\frac{1}{2}-\frac{1}{2}\frac{(r-\sigma^2/2)}{\sigma}\sqrt{\Delta t} \qquad \text{Eq. 5b}$$

Let's define

$$z=\frac{i-Np}{\sqrt{Npq}}$$

And take the limit $N\to\infty$

$$\sum_{i=k}^{N} \to \int_{k'}^{\infty}\sqrt{Npq}\,dz$$

And I_2 becomes

$$I_2=\int_{k'}^{\infty}dz\sqrt{Npq}\,e^{-z^2/2}=\int_{k'}^{\infty}dz\frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

Where k' is

$$k'=\frac{k-Np}{\sqrt{Npq}}=\frac{\frac{\ln(X/S_0)}{2\sigma\sqrt{\Delta t}}+\frac{N}{2}-Np}{\sqrt{Npq}}$$
$$=\frac{\ln(X/S_0)}{2\sigma\sqrt{N\Delta t}\sqrt{pq}}+\sqrt{N}\,\frac{(1-2p)}{2\sqrt{pq}}$$

From Eqs. 5a and 5b

$$\sqrt{pq}=\sqrt{\frac{1}{4}-\frac{1}{4}\frac{(r-\sigma^2/2)^2}{\sigma^2}\Delta t}\underset{\Delta t\to 0}{\to}\frac{1}{2} \qquad \text{Eq. 6}$$

And

$$\sqrt{N}\,(1-2p) = -\frac{(r-\sigma^2/2)^2}{\sigma}\sqrt{N\Delta t}\ ,\ N\Delta t \to T$$

So,

$$k' = \frac{\ln(X/S_0)}{\sigma\sqrt{T}} - \frac{(r-\sigma^2/2)}{\sigma}\sqrt{T}$$

And

$$I_2 = \int_{k'}^{\infty} dz \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

With a change of variables $z \to -z'$

$$I_2 = \int_{-\infty}^{-k'} dz\ \frac{e^{-z^2/2}}{\sqrt{2\pi}} = N(\frac{\ln(S_0/X)}{\sigma\sqrt{T}} + \frac{(r-\sigma^2/2)}{\sigma}\sqrt{T}) = N(d_2)$$

Where

$$d_2 = \frac{\ln(S_0/X)}{\sigma\sqrt{T}} + \frac{(r-\sigma^2/2)}{\sigma}\sqrt{T}$$

Next, we compute I_1.

$$I_1 = \sum_{i=k}^{N} u^i\, d^{N-i}\, \frac{N!}{i!(N-i)!} p^i\, q^{N-i}$$

Using Eq. 4 and $d = u^{-1}$

$$u^i d^{N-i} = u^{-N+2i} = e^{(2i-N)\sigma\sqrt{\Delta t}}$$

$$I_1 = \sum_{i=k}^{N} e^{(2i-N)\sigma\sqrt{\Delta t}}\ \frac{1}{\sqrt{2\pi Npq}} e^{-(i-Np)^2/2Npq}$$

We can rewrite $e^{(2i-N)\sigma\sqrt{\Delta t}}\ e^{-(i-Np)^2/2Npq}$ as

$$\exp\left\{\left[-\frac{(i-Np(1+2\sigma\sqrt{\Delta t}q))^2}{2Npq}\right] + 2Np\sigma\sqrt{\Delta t} + 2Npq\sigma^2\Delta t - N\sigma\sqrt{\Delta t}\right\}$$

Now define

$$z = \frac{i - Np(1+2\sigma\sqrt{\Delta t}\ q)}{\sqrt{Npq}}$$

And

$$\Delta z = \Delta i / \sqrt{Npq}$$

And also note that

$$\begin{aligned}2Np\sigma\sqrt{\Delta t} - N\sigma\sqrt{\Delta t} &\approx N\sigma\sqrt{\Delta t}\,(2p-1) \\ &= N\sigma\sqrt{\Delta t}\,\frac{(r-\sigma^2/2)}{\sigma}\sqrt{\Delta t} \\ &= N\Delta t\,(r-\sigma^2/2) = T\,(r-\sigma^2/2)\end{aligned}$$

And with

$$pq \approx \frac{1}{4}$$

We get

$$2Npq\sigma^2\Delta t \approx 2N\left(\frac{1}{4}\right)\sigma^2\Delta t = \frac{N}{2}\Delta t\sigma^2 = T\sigma^2/2$$

Using

$$\sum_{i=k}^{N} \;\to \int_{k'}^{\infty} \sqrt{Npq}\, dz$$

$$I_1 = \int_{k''}^{\infty} dz \frac{\sqrt{Npq}}{\sqrt{2\pi Npq}} e^{-z^2/2}\, e^{T(r-\sigma^2/2)}\, e^{T\sigma^2/2}$$

$$= e^{rT} \int_{k''}^{\infty} dz \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

Where

$$k'' = \frac{k - Np(1+2\sigma\sqrt{\Delta t}\, q)}{Npq}$$

Inserting k from Eq. 2

$$k'' = \frac{\ln(X/S_0)/2\sigma\sqrt{\Delta t} + N/2 - Np(1+2\sigma\sqrt{\Delta t}\,q)}{\sqrt{Npq}}$$

$$= \frac{\ln(X/S_0)}{\sigma\sqrt{T}} + \frac{\sqrt{N}}{\sqrt{pq}}\left(\frac{1}{2} - p - 2p\sigma\sqrt{\Delta t}\,q\right)$$

Now take the limit $N\Delta t \underset{\substack{N\to\infty \\ \Delta t\to 0}}{\to} T$ and use Eq. 5a for p and $pq \approx \frac{1}{4}$; then, the second term in the equation above can be rewritten as

$$\frac{\sqrt{N}}{\sqrt{pq}}\left(\frac{1}{2} - p - 2p\sigma\sqrt{\Delta t}\,q\right) = -\frac{1}{2}\frac{(r-\sigma^2/2)}{\sigma}\sqrt{\Delta t} - \frac{\sigma\sqrt{\Delta t}}{2} = -\frac{1}{2}\frac{(r+\sigma^2/2)}{\sigma}\sqrt{\Delta t}$$

k'' reduces to

$$k'' = \frac{\ln(X/S_0)}{\sigma\sqrt{T}} - \frac{(r+\sigma^2/2)}{\sigma}\sqrt{T}$$

Now I_1 can be rewritten as

$$I_1 = e^{rT}\int_{k''}^{\infty} dz\, e^{-z^2/2}/\sqrt{2\pi}$$

With a change of variable $z \to -z$ and defining d_1 as

$$d_1 = \frac{\ln(S_0/X)}{\sigma\sqrt{T}} + \frac{(r+\sigma^2/2)}{\sigma}\sqrt{T} = -k''$$

I_1 now becomes

$$I_1 = e^{rT}\int_{-\infty}^{d_1} dz\, e^{-z^2/2}/\sqrt{2\pi}$$

Now that we have expressions for I_1 and I_2, we can write down the expression for the price of a call option, C_0, as follows:

$$C_0 = e^{-rT} S_0 e^{rT} N(d_1) - X N(d_2) e^{-rT}$$
$$= S_0 N(d_1) - X N(d_2) e^{-rT}$$